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## NOTE ON SCHLÄFLI'S ELLIPTIC MODULAR EQUATIONS.

BY ARTHUR BERRY.

In a former paper in this journal\* I proved some properties of the elliptic modular equations substantially in the form commonly known as Schläfli's equations. I worked with a modular function  $x(\tau) = 2^{-1/6}\chi(\tau)$ , where  $\chi$  is Hermite's function† and showed that, for transformations of prime order  $n$  ( $> 3$ ), (1) when  $n$  is of the form  $4p - 1$ , corresponding to  $\tau = i$ ,  $x = 2^{-1/4}$  the roots of the modular equations are equal in pairs and branched, each pair corresponding to a branch point of order 1 on the corresponding Riemann surface and that (2) when  $n$  is of form  $4p + 1$ , there are  $n - 1$  roots equal in pairs and branched, and two isolated roots. I showed further that the two isolated roots are always of the form  $\epsilon^\lambda x(i)$  ( $\lambda$  an integer,  $\epsilon = e^{2i\pi/24}$ ), and that for  $n = 8p - 3$ , the two values of  $\epsilon^\lambda$  are  $\pm i$  ( $\lambda = \pm 6 \bmod{24}$ ), while for  $n = 8p + 1$  both values are  $-1$ , or both values  $1$ , but was unable to give any simple criterion distinguishing these last two cases.

The object of this note is to establish these results as to the isolated roots in a somewhat simpler way, avoiding the rather troublesome quadratic transformation used before, and to distinguish between the last two cases.

It is known that for a modular substitution  $\{(c + d\tau)/(a + b\tau), \tau\}$  of Hermite's first type ( $a, d$  odd,  $b, c$  even)

$$x\{(c + d\tau)/(a + b\tau)\} = \epsilon^\lambda x(\tau),$$

where  $\lambda = \frac{1}{2}(b - c)(bcd - a)$ ;‡ a similar equation holds for substitutions of the second type, but as these can be derived from those of the first type by applying the substitution  $T(T\tau = -1/\tau)$  and we are only concerned with  $\tau = i$ , so that  $T\tau = \tau$ , it is enough to consider substitutions of the first type. Hence in order to prove that  $x\{(48r + i)/n\} = \epsilon^\lambda x(i)$ , it is enough to prove that it is possible to find integers  $a, b, c, d$ , where  $a, d$  are odd,  $b, c$  even and  $ad - bc = 1$ , and an integer  $r$  such that

$$(48r + i)/n = (c + di)/(a + bi). \quad (1)$$

If  $n$  is a prime number of the form  $4p + 1$ , it is a well known result that

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\* On Elliptic Modular Equations for Transformations of Orders 29, 31, 37; Vol. XXX, pp. 156-169.

† Sur la résolution de l'équation du quatrième degré; *Comptes Rendus*, Vol. 46 (1858), *Oeuvres*, Vol. II, p. 28.

‡ Tannery and Molk, *Fonctions Elliptiques*, Vol. II, Table XLVI.

we can choose  $a$  and  $b$  ( $a$  odd,  $b$  even) so that  $a^2 + b^2 = n$ .<sup>\*</sup> With this choice of  $a$  and  $b$  (1) is equivalent to

$$ac + bd = 48r, \quad (2)$$

$$ad - bc = 1. \quad (3)$$

We can now choose  $c, d$  to satisfy (3) and can further arrange so that  $c$  is even  $d$  odd; if  $c', d'$  is one such solution, the general solution is  $c = c' + 2ka$ ,  $d = d' + 2kb$  ( $k$  integral), and (2) can be satisfied, if

$$2k(a^2 + b^2) + ac' + bd' \equiv 0, \quad \text{mod } 48,$$

or

$$kn \equiv -\frac{1}{2}(ac' + bd'), \quad \text{mod } 24;$$

since  $n$  is prime and  $ac' + bd'$  is even, this congruence can be satisfied;  $a, b, c, d$  are now found, and  $r$  is given by (2).

It remains to determine the integer  $\lambda$  (*mod.* 24), which depends on the congruences *mod.* 3 and *mod.* 16, satisfied by  $a, b, c, d$ .

From (2) and (3), it follows at once that if any one of  $a, b, c, d \equiv 0$ , *mod.* 3, then either  $a$  and  $d$  or  $b$  and  $c$  satisfy this congruence and therefore also  $\lambda \equiv 0$ , *mod.* 3; if no one of  $a, b, c, d \equiv 0$ , then from (3)  $ad \equiv -1$ ,  $bc \equiv 1$ , whence  $b - c \equiv 0$ , so that again  $\lambda \equiv 0$ . Thus in all cases  $\lambda \equiv 0$ , *mod.* 3.

The congruences *mod.* 16 are rather more troublesome. From (2) it follows at once that if  $b = 2^k \times (\text{odd integer})$ ,  $c = 2^{k'} \times (\text{odd integer})$ , then  $k' = k$ , for  $k = 1, 2, 3$ , and  $k' \geq 4$  for  $k \geq 4$ ; hence for  $k \geq 2$  (which is the same condition as  $b \equiv 0$ , *mod.* 8)  $b - c \equiv 0$  *mod.* 16 and  $\lambda \equiv 0$ , *mod.* 8; if  $k = 1$  or 2,  $ad = bc + 1 \equiv 1$ , *mod.* 4 and  $a \equiv d \equiv \pm 1$ , *mod.* 4, so that  $a + d \equiv 2$ , *mod.* 4. We now have  $a(b - c) = (a + d)b - (ac + bd) \equiv (a + d)b$ , *mod.* 16, by (2), whence  $b - c = 2^{k+1} \times (\text{odd integer})$ , so that for  $k = 1$ ,  $b - c \equiv \pm 4$ , *mod.* 16,  $\lambda \equiv \pm 2$ , *mod.* 8, and for  $k = 2$ ,  $b - c \equiv 8$ , *mod.* 16,  $\lambda \equiv 4$ , *mod.* 8. As we have seen that  $\lambda \equiv 0$ , *mod.* 3 it follows that  $\lambda = \pm 6, 12, 0$ , *mod.* 24 and  $\epsilon^\lambda = \pm i, -1, 1$  according as  $k = 1, k = 2, k > 2$ . The condition  $k = 1$  (or  $b \equiv 2$ , *mod.* 4) can be replaced by the simpler condition  $n \equiv -3$ , *mod.* 8, for  $a^2$  being the square of an odd number is necessarily of the form  $8p + 1$ , so that  $b^2 = n - a^2 \equiv n - 1$  *mod.* 8, and then  $b \equiv 2$ , *mod.* 4 or  $b \equiv 0$  *mod.* 4 according as  $n \equiv -3, 1$ , *mod.* 8.

The discrimination between the cases  $k = 2, k > 2$  (or  $b \equiv 4, b \equiv 0$ , *mod.* 8), does not appear possible by means of any linear congruence for  $n$  but requires the actual expression of  $n$  ( $n = 8p + 1$ ) in the form  $a^2 + b^2$  or some equivalent process in the arithmetical theory of quadratic forms.

<sup>\*</sup> Mathews, *Theory of Numbers*, § 91.

By the proof already given if one isolated root is known the other is its conjugate imaginary, so that if one is  $\pm ix(i)$ ,  $= \pm i2^{-1/4}$  the other is  $\mp ix(i)$ , and if one is  $\pm x(i)$  the other is equal to it.

Thus we have the final result:

If  $n \equiv -1 \pmod{4}$ , all the roots correspond to branch points; if  $n \equiv -3 \pmod{8}$ , there are two isolated roots  $\pm i2^{-1/4}$ ; if  $n \equiv 1 \pmod{8}$ , there are two isolated roots both  $-2^{-1/4}$  or both  $2^{1/4}$  according as, when  $n$  is expressed in the form  $a^2 + b^2$  ( $a$  odd,  $b$  even),  $b \equiv 4$  or  $b \equiv 0 \pmod{8}$ .

I take the opportunity of correcting some errata in my former paper:

P. 158, line 11, *for mod. 48 read mod. 24.*

P. 165, line 18, *for  $x\{(-1 + i\sqrt{27})/31\}$  read  $x\{(-2 + i\sqrt{27})/31\}$ .*

P. 165, line 23, *for 24 read  $2^4$ .*

P. 165, lines 23, 26, *for  $\sqrt{19}$  read  $\sqrt{15}$ .*

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